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A NOTE ON THE CHARACTERISTIC RANK AND RELATED NUMBERS

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Dedicated to Professor Masaharu Morimoto on the occasion of his 60th birthday

Abstract. This note quantifies, via a sharp inequality, an interplay between (a) the characteristic rank of a vector bundle over a topological space X, (b) the $\mathbb{Z}_2$-Betti numbers of X, and (c) sums of the numbers of certain partitions of integers. In a particular context, (c) is transformed into a sum of the readily calculable Betti numbers of the real Grassmann manifolds.

1. Introduction

The *characteristic rank* of a smooth closed connected d-dimensional manifold M was introduced, and in some cases also calculated, in [3] as the largest integer k, $0 \leq k \leq d$, such that each cohomology class in $H^j(M;\mathbb{Z}_2)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of M; further results can be found in [2]. Recently, A. Naolekar and A. Thakur ([6]) have adapted this homotopy invariant of smooth closed connected manifolds to vector bundles. Given a path-connected topological space X and a real vector bundle α over X, by the *characteristic rank of* α, denoted $\mathrm{charrank}(\alpha)$, we understand the largest k, $0 \leq k \leq \dim_{\mathbb{Z}_2}(X)$, such that every cohomology class in $H^j(X;\mathbb{Z}_2)$, $0 \leq j \leq k$, can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(\alpha) \in H^i(X;\mathbb{Z}_2)$; $\dim_{\mathbb{Z}_2}(X)$ is the supremum of all q such that the cohomology groups $H^q(X;\mathbb{Z}_2)$ do not vanish. We shall always use $\mathbb{Z}_2$ as the coefficient group for cohomology, and so we shall write $H^i(X)$ instead of $H^i(X;\mathbb{Z}_2)$ and $\dim(X)$ instead of $\dim_{\mathbb{Z}_2}(X)$. Of course, if TM denotes the tangent bundle of M, then we have $\mathrm{charrank}(TM) = \mathrm{charrank}(M)$. Results on the characteristic rank of vector bundles over the Stiefel manifolds can be found in [4]. In addition to being an interesting question in its own right, there are other reasons for investigating the characteristic rank; one of them is its close relation to the cup-length of a given space ([3], [6]).

This note presents a result on the characteristic rank that may also prove useful in some non-topological questions (for instance, in the theory of partitions; see [1]). More precisely, we quantify (Theorem 2.1), via a sharp inequality, an interplay between (a) the characteristic rank of a vector bundle, (b) the $\mathbb{Z}_2$-Betti numbers of base space of this vector bundle, and (c) sums of the numbers of certain partitions of integers. In a particular context, (c) is transformed into a sum of the readily

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calculable Betti numbers of the real Grassmann manifolds $G_{n,k}$ of all k-dimensional vector subspaces in $\mathbb{R}^n$.

2. The result and its proof

Let $p(a,b,c)$ be the number of partitions of c into *at most* b parts, each $\leq a$. At the same time, given a subset A of the positive integers, let $p(A,b,c)$ denote the number of partitions of c into b parts each taken from the set A. We recall that ([5, 6.7]) $p(a,b,c)$ is the same as the number of cells of dimension c in the Schubert cell decomposition of the Grassmann manifold $G_{a+b,b}$, or the same as the $\mathbb{Z}_2$-Betti number $b_c(G_{a+b,b}) = b_c(G_{a+b,a})$. Of course, for dimensional reasons, $b_c(G_{a+b,b}) = p(a,b,c) = 0$ for $c > ab = \dim(G_{a+b,b})$. In addition, we denote by $p(c)$ the total number of partitions of c. Even if not explicitly stated, X will always mean a path-connected topological space.

For obvious reasons, we confine our considerations to those vector bundles having total Stiefel-Whitney class non-trivial. For a given space X, an ordered subset in $\{1,2,\ldots,\dim(X)\}$, with the least element (denoted by) ν and the greatest element (denoted by) κ, will be denoted by S_ν^κ. If, in addition, α is a real vector bundle over X, then $S_\nu^\kappa(\alpha)$ will denote any $S_\nu^\kappa \neq \emptyset$ such that $w_i(\alpha) = 0$ for every positive $i \notin S_\nu^\kappa$. In general, there are several possible choices of $S_\nu^\kappa(\alpha)$: one can always take $S_\nu^\kappa(\alpha) = \{1=\nu, 2, \ldots, \dim(X) = \kappa\}$, but if we know, for instance, that $w_3(\alpha) = 0$, then we can also take the set $\{1 = \nu, 2, 4, \ldots, \dim(X) = \kappa\}$ in the role of $S_\nu^\kappa(\alpha)$. Now we can state and prove our result.

Theorem 2.1. *Let α be a real vector bundle over a path-connected topological space X such that $w(\alpha) \neq 1$, and let* $\operatorname{charrank}(\alpha) \geq t$ *for some t. Then, for every $j \in \{1,\ldots,t\}$, we have an inequality,*

$$b_j(X) \leq \sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{x \in S_\nu^\kappa(\alpha) | x \leq \mu\}, s, j), \tag{1}$$

where $\mu = \min\{j, \kappa\}$.

In particular, if the set $\{x \in S_\nu^\kappa(\alpha) | x \leq \mu\}$ in (1) is gapless, then (1) turns into

$$b_j(X) \leq \sum_{\frac{j}{\mu} \leq s \leq \frac{j}{\nu}} b_{j-\nu s}(G_{\mu-\nu+s,s}). \tag{2}$$

Proof. Let us fix one of the sets $S_\nu^\kappa(\alpha)$. If $\operatorname{charrank}(\alpha) \geq t$ for some t then, for every $j \in \{1,2,\ldots,t\}$, the $\mathbb{Z}_2$-vector space $H^j(X)$ is spanned by all the products of the form

$$w_\nu^{i_\nu}(\alpha) \cup \cdots \cup w_\mu^{i_\mu}(\alpha) \in H^j(X). \tag{3}$$

Since $\nu(i_\nu + \cdots + i_\mu) \leq \nu i_\nu + \cdots + \mu i_\mu = j$ (thus $i_\nu + \cdots + i_\mu \leq \frac{j}{\nu}$), the number of generators of the form (3), that is, an upper bound for $b_j(X)$, is the number of partitions of j into at most $\lfloor \frac{j}{\nu} \rfloor$ positive parts each taken from the set $\{x \in S_\nu^\kappa(\alpha) | x \leq \mu\}$. In other words, this upper bound is $\sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{x \in S_\nu^\kappa(\alpha)|x \leq \mu\}, s, j)$ as was asserted.

In order to transform (1) into (2) when the set $\{x \in S_\nu^\kappa(\alpha) | x \le \mu\}$ in (1) is gapless, it suffices to know that

$$\sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\{\nu, \nu+1, \nu+2, \ldots, \mu\}, s, j) = \sum_{s=1}^{\lfloor \frac{j}{\nu} \rfloor} p(\mu-\nu, s, j-\nu s); \tag{4}$$

this equality is verified by the following elementary considerations.

Let $P(j)^x_{\{\nu,\nu+1,\ldots,\mu\}}$ be the set of partitions of j into x positive parts each taken from the set $\{\nu, \nu+1, \ldots, \mu\}$ and let $P(l)^{\le x}_{\{1,2,\ldots,\mu-\nu\}}$ be the set of partitions of l ($l \ge 0$) into at most x parts each taken from the set $\{1, 2, \ldots, \mu-\nu\}$ (the set $P(0)^{\le x}_{\{1,2,\ldots,\mu-\nu\}}$ for all $x > 0$ and $\mu - \nu > 0$ contains just one element, namely the empty partition). Of course, the total number of elements in $P(l)^{\le x}_{\{1,2,\ldots,\mu-\nu\}}$ is $p(\mu-\nu, x, l)$.

Each element of $P(j)^x_{\{\nu,\nu+1,\ldots,\mu\}}$ has the form

$$a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x,$$

where $i(\nu) \ge 0$, $a_1 = \cdots = a_{i(\nu)} = \nu < a_{i(\nu)+1} \le a_{i(\nu)+2} \le \cdots \le a_x$ and $a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x = j$. The map

$$P(j)^x_{\{\nu,\nu+1,\ldots,\mu\}} \longrightarrow P(j-\nu x)^{\le x}_{\{1,2,\ldots,\mu-\nu\}},$$

$$a_1 + \cdots + a_{i(\nu)} + a_{i(\nu)+1} + \cdots + a_x \mapsto (a_{i(\nu)+1} - \nu) + (a_{i(\nu)+2} - \nu) + \cdots + (a_x - \nu)$$

is bijective; indeed, the inverse map is readily seen to be

$$P(j-\nu x)^{\le x}_{\{1,2,\ldots,\mu-\nu\}} \longrightarrow P(j)^x_{\{\nu,\nu+1,\ldots,\mu\}},$$

$$b_1 + b_2 + \cdots + b_{x-i} \mapsto \underbrace{\nu + \cdots + \nu}_{i \text{ times}} + (b_1 + \nu) + \cdots + (b_{x-i} + \nu).$$

Thus (4) is verified, and the proof of Theorem 2.1 is finished. □

Example 2.2. We recall ([5, Theorem 7.1]) that the cohomology algebra of the infinite Grassmannian $G_{\infty,k}$ can be identified with a polynomial algebra,

$$H^*(G_{\infty,k}) = \mathbb{Z}_2[w_1, \ldots, w_k],$$

where $w_i \in H^i(G_{\infty,k})$ is the ith Stiefel-Whitney class of the universal k-plane bundle $\gamma_{\infty,k}$. Thus $\operatorname{charrank}(\gamma_{\infty,k}) = \infty$ and we may take $S_\nu^\kappa(\gamma_{\infty,k}) = \{1 = \nu, 2, \ldots, k = \kappa\}$. Since for $X = G_{\infty,k}$ there are no relations among the generators of the form (3), inequalities (1) and (2) turn into one of the following equalities for any positive integer j:

$$b_j(G_{\infty,k}) = p(j) = \sum_{s=1}^{j} b_{j-s}(G_{j-1+s,s}) \text{ if } j \le k, \tag{5}$$

$$b_j(G_{\infty,k}) = \sum_{s=1}^{j} p(\{1,2,\ldots,k\}, s, j) = \sum_{s=\lceil \frac{j}{k} \rceil}^{j} b_{j-s}(G_{k-1+s,s}) \text{ if } j > k. \tag{6}$$

In a similar way, one can see that (1) and (2) are also sharp for $X = \tilde{G}_{\infty,k}$, the infinite oriented Grassmannian.

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